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Del Pezzo surfaces and affine 7-brane backgrounds

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Abstract: A map between string junctions in the affine 7-brane backgrounds and vector bundles on Del Pezzo surfaces is constructed using mirror symmetry. It is shown that the lattice of string junctions with support on an affine 7-brane configuration is isomorphic to the K-theory group of the corresponding Del Pezzo surface. This isomorphism allows us to construct a map between the states of the $\mathcal{N}=2$, $D=4$ theories with E_N global symmetry realized in two different ways in type-IIB and type-IIA string theory. A subgroup of the $\mathrm{SL}(2,\mathbb{Z})$ symmetry of the $\widehat{E}_9$ 7-brane background appears as the Fourier-Mukai transform acting on the D-brane configurations realizing vector bundles on elliptically fibered $\mathcal{B}_9$.

Keywords: D-branes, String Duality.

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Contents

1. Introduction

The study of D-branes has provided important insights into non-perturbative aspects of string theory as well as supersymmetric gauge theories. Supersymmetric field theories are realized in string theory either as the effective world-volume theory of a configuration of D-branes or as the theory on the transverse space after compactification on some appropriate manifold. In the later case D-branes wrapped on various cycles of the compactification manifold give rise to particles and non-critical strings in the gauge theory. Thus D-branes provide a geometrical description of states in the supersymmetric gauge theories.

Along the process of realizing QFT's as effective models in string theory, previously unknown exotic SUSY theories have also been found. An example is the $\mathcal{N} = 1$ six dimensional E_8 theory [1]–[7] whose compactification on a torus leads to $\mathcal{N} = 2$ theories with E_N global symmetry [8]. These four dimensional theories and their spectra are the focus of our paper. Since there are different ways of obtaining effective field theories in string theory, some models can be realized in different ways. Our aim is to compare two different realizations of the same theory, namely $d = 4$, $\mathcal{N} = 2$ SYM theory with E_N global symmetry which arises both as the worldvolume theory of a threebrane in IIB (or F-theory) [9, 10, 11, 12] near a particular set of 7-branes, and as the compactification of IIA on a CY threefold [12, 13, 14, 7, 15] with a shrinking Del Pezzo surface in it. In this paper we will construct the precise map between the spectra of the two different realizations of the theory.

We will first review both constructions. The states in the IIA theory on the transverse space are obtained by wrapping D-branes on various cycles in the Del Pezzo and the description of the spectrum consists of the classification of sheaves on the surface. On the type-IIB side the states of the D3-brane world-volume theory are (p, q) strings and string junctions with support on 7-branes and the D3-brane. In the next section we explain that both the image of the sheaves in the K-theory group and the junctions form a lattice. In the third section an isomorphism between these two lattices is established and in the last section an application of this isomorphism is given which identifies the Fourier-Mukai transform [16] of sheaves on $\mathcal{B}_9$ with the SL$(2, \mathbb{Z})$ symmetry of the dual brane configuration in IIB.

2. $\mathcal{N} = 2$ theories with E_N global symmetry

2.1 Compactification of IIA

We begin by reviewing how $d = 4$, $\mathcal{N} = 2$ field theories with exceptional global symmetry can be be realized in IIA string theory. Of the several ways involving the low energy limit of D-brane configurations or compactifications we use the approach commonly referred to as geometric engineering. Compactification of IIA string theory on a non-compact CY threefold with a shrinking 4-cycle $\mathcal{X}$ gives rise to a low energy theory with 8 supercharges on the transverse space, whose spectrum is determined by the homology of that 4-cycle. When the shrinking manifold is a Del Pezzo surface, the low energy theory acquires exceptional global symmetry because the lattice of 2-cycles contains the root lattice of an exceptional algebra and thus admits the action of the corresponding Weyl group.

The states of the field theory are obtained from IIA D-branes wrapping various submanifolds of $\mathcal{X}$: D0's on points, D2's wrapping 2-cycles and D4's on $\mathcal{X}$ itself. A configuration of Q D4-branes is a $U(Q)$-bundle on $\mathcal{X}$ with instanton number given by the number of D0's and first Chern class being the Poincaré dual to the homology class wrapped by D2-branes [17, 18]. Proper description of these D-brane configurations requires the more sophisticated notion of sheaves on $\mathcal{X}$ [19, 20]. This becomes essential if configurations with and without D4-branes are to be dealt with on equal footing. BPS states arise from supersymmetric configurations of D-branes.

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Wrapped D4-branes with immersed lower dimensional branes correspond to semi-stable torsion-free sheaves on $\mathcal{X}$ [19][1] while a generic configuration is described by a coherent sheaf which is reducible to torsion and semi-stable torsion-free components corresponding to non-immersed branes.

A Del Pezzo surface $\mathcal{X}$ is a two complex dimensional manifold constructed by blowing up N points on $\mathbb{P}^2$ or $N-1$ points on $\mathbb{P}^1 \times \mathbb{P}^1$, where $N \leq 8$. We denote these two families as $\widetilde{\mathcal{B}}_N$ and $\mathcal{B}_N$, respectively; moreover $\widetilde{\mathcal{B}}_N = \mathcal{B}_N$ for $N > 1$, and thus it is sufficient to consider $\mathcal{B}_1 = \mathbb{P}^1 \times \mathbb{P}^1$ and $\widetilde{\mathcal{B}}_N$ for $N = 0, \ldots, 8$. The almost Del Pezzo $\widetilde{\mathcal{B}}_9$ is also known as $\frac{1}{2}$K3 [21] since it is an elliptically fibered manifold with a base $B \cong \mathbb{P}^1$ and twelve degenerate elliptic fibers.

2.1.1 The homology of $\widetilde{\mathcal{B}}_N$ and the E_N root lattice

The case $\widetilde{\mathcal{B}}_{N\leq 8}$: the 2nd homology group $H_2(\widetilde{\mathcal{B}}_N)$ is $N+1$ dimensional and is generated by the elements $\{l, e_1, \ldots, e_N\}$, where l is the generator of $H_2(\mathbb{P}^2)$ and e_i $(i = 1, \ldots, N)$ are the exceptional curves. The intersection numbers are

$$^{\#}(l \cdot l) = 1\,, \qquad ^{\#}(e_i \cdot e_j) = -\delta_{ij}\,, \qquad ^{\#}(l \cdot e_i) = 0\,. \tag{2.1}$$

The canonical class is $K_{\widetilde{\mathcal{B}}_N} = -3l + \sum_{i=1}^N e_i$, it is used to define the degree of a 2-cycle Σ as

$$\mathrm{d}_\Sigma \equiv -^{\#}(K_{\widetilde{\mathcal{B}}_N} \cdot \Sigma)\,, \qquad \Sigma \in H_2(\widetilde{\mathcal{B}}_N)\,. \tag{2.2}$$

The homology lattice $H_2(\widetilde{\mathcal{B}}_N)$ contains the root lattice Γ_N of the E_N algebra. To see this, identify the set of roots as $\Delta_N = \{C \in H_2(\widetilde{\mathcal{B}}_N, \mathbb{Z}) | ^{\#}(C \cdot C) = -2, \mathrm{d}_C = 0\}$, and choose the simple roots of $E_N, \{3 \leq N \leq 9\}$ as

$$C_i \equiv e_i - e_{i+1}\,, \quad i = 1, \ldots, N-1 \qquad \text{and} \qquad C_N \equiv l - e_1 - e_2 - e_3\,, \tag{2.3}$$

their intersection numbers yield the E_N Cartan matrix. Using C_i as basis elements, we can define the weight vector $\{\omega^i | i = 1, \ldots, N\}$ and associate Dynkin labels λ_i with each curve:

$$^{\#}(\omega^i \cdot C_j) = -\delta^i_j\,, \qquad \lambda_i \equiv -^{\#}(C \cdot C_i)\,, \qquad i = 1, \ldots, N\,, \tag{2.4}$$

such that the curve and the self-intersection is given in terms of the corresponding weight vector:

$$\begin{aligned} \Sigma &= \sum_{i=1}^N \lambda_i \omega^i - \frac{\mathrm{d}_\Sigma}{9-N} K_{\widetilde{\mathcal{B}}_N}\,, \\ ^{\#}(\Sigma \cdot \Sigma) &= -\vec{\lambda} \cdot \vec{\lambda} + \frac{\mathrm{d}_\Sigma^2}{9-N}\,, \qquad \Sigma \in H_2(\widetilde{\mathcal{B}}_{N\leq 8}, \mathbb{Z})\,. \end{aligned} \tag{2.5}$$

[1] In this case the sheaf is constructed from the sections of the holomorphic bundle.

The case $\widetilde{\mathcal{B}}_9$: blowing up one more point we arrive at the final case we consider: $\widetilde{\mathcal{B}}_9 = \frac{1}{2}K3$ which is not strictly speaking Del Pezzo. It is however often referred to as almost Del Pezzo having a nef (but not ample) anticanonical class. Since $\widetilde{\mathcal{B}}_9$ is elliptically fibered we will use a different basis for $H_2(\widetilde{\mathcal{B}}_9)$. We denote the homology class of the base and the fiber by B and F respectively. With the choice of basis $\{C_1 \cdots C_8, B+F, B\}$, $H_2(\widetilde{\mathcal{B}}_9) = \Gamma_{E_8} \oplus \left(\begin{smallmatrix} 1 & 0 \\ 0 & -1 \end{smallmatrix}\right)$. In terms of the degree d_Σ and $\mathrm{c} = {}^\#(B \cdot \Sigma)$ the curve Σ and its self-intersection number is given by

$$\begin{aligned} \Sigma &= \sum_{i=1}^{8} \lambda_i \omega^i + \mathrm{d}_\Sigma (B+F) + \mathrm{c}F\,, \\ {}^\#(\Sigma \cdot \Sigma) &= -\vec{\lambda} \cdot \vec{\lambda} + \mathrm{d}_\Sigma^2 + 2\mathrm{c}\mathrm{d}_\Sigma\,, \qquad \Sigma \in H_2(\widetilde{\mathcal{B}}_9, \mathbb{Z})\,. \end{aligned} \tag{2.6}$$

Since $H_2(\widetilde{\mathcal{B}}_9)$ contains the affine E_8 root lattice, it admits the action of the affine E_8 Weyl group and all curves fall into representations of affine E_8 such that d_Σ and c are the level and the grade of the representation respectively.

2.1.2 The K-theory group of Del Pezzo surfaces

The K-theory group of $\widetilde{\mathcal{B}}_N$ and $\mathcal{B}_1$ are given as [22]:

$$\mathrm{K}(\widetilde{\mathcal{B}}_N) = \underbrace{\mathbb{Z} \oplus \cdots \oplus \mathbb{Z}}_{N+3}, \qquad \mathrm{K}(\mathcal{B}_1) = \mathbb{Z} \oplus \mathbb{Z} \oplus \mathbb{Z} \oplus \mathbb{Z}\,. \tag{2.7}$$

The K-theory group K^0 of $\widetilde{\mathcal{B}}_N$ ($\mathcal{B}_1$) is generated by $N+3$ (4) elements and being torsion free it is isomorphic to the even-degree integral cohomology group $H^*(\widetilde{\mathcal{B}}_N)$ ($H^*(\mathcal{B}_1)$). The map is given by

$$\mathrm{K}(\mathcal{X}) \ni \mathcal{E} \longrightarrow \mathrm{ch}(\mathcal{E}) \in H^*(\mathcal{X})\,. \tag{2.8}$$

There is a natural bilinear form on $\mathrm{K}(\mathcal{X})$ [22]: let $\mathcal{E}_{1,2} \in \mathrm{K}(\mathcal{X})$ then

$$\begin{aligned} \langle \mathcal{E}_1, \mathcal{E}_2 \rangle &\equiv \int_X \mathrm{ch}(\mathcal{E}_1 \otimes \mathcal{E}_2^*) \wedge \mathrm{Td}(\mathcal{X}) \\ &= \int_X \mathrm{ch}(\mathcal{E}_1) \wedge \mathrm{ch}(\mathcal{E}_2^*) \wedge \mathrm{Td}(\mathcal{X}) \equiv \int_X \mathrm{ch}(\mathcal{E}_1) \wedge \mathrm{ch}(\mathcal{E}_2)^\vee \wedge \mathrm{Td}(\mathcal{X})\,, \end{aligned} \tag{2.9}$$

where $\mathcal{E}^*$ is the dual bundle, $\mathrm{Td}(\mathcal{X}) = 1 + \frac{1}{2}c_1(\mathcal{X}) + \frac{1}{12}(c_1(\mathcal{X})^2 + c_2(\mathcal{X}))$ and if $v = \sum_{i=0}^2 v_i$, $v_i \in H^{2i}(\mathcal{X})$ then $v^\vee \equiv \sum_{i=0}^2 (-1)^i v_i$. Let us write[2] the Chern classes of $\mathcal{E}_{1,2} \in \mathrm{K}(\mathcal{X})$ as $\mathrm{ch}(\mathcal{E}_a) = (\mathrm{r}_a, \Sigma_a, \mathrm{ch}_2(\mathcal{E}_a))$; then we obtain[3]

$$\langle \mathcal{E}_1, \mathcal{E}_2 \rangle = \mathrm{r}_1\mathrm{r}_2 - {}^\#(\Sigma_1 \cdot \Sigma_2) + \mathrm{r}_1\,\mathrm{ch}_2(\mathcal{E}_2) + \mathrm{r}_2\,\mathrm{ch}_2(\mathcal{E}_1) + \frac{1}{2}(\mathrm{r}_2\mathrm{d}_{\Sigma_1} - \mathrm{r}_1\mathrm{d}_{\Sigma_2})\,. \tag{2.10}$$

An element (r, Σ, ch_2) $\in H^*(\mathcal{X})$ represents an equivalence class $\mathcal{E}$ of sheaves on $\mathcal{X}$ such that $\mathrm{ch}(\mathcal{E})$=(r, Σ, $\mathrm{ch}_2(\mathcal{E})$) where $\mathrm{ch}_2(\mathcal{E}) = \frac{1}{2}{}^\#(\Sigma \cdot \Sigma) - \int_{\mathcal{X}} c_2(\mathcal{E})$.

[2] We will use the same symbol for the 2-form and its dual 2-cycle. Thus ${}^\#(\Sigma_a \cdot \Sigma_b) \equiv \int_{\mathcal{X}} \Sigma_a \wedge \Sigma_b$.

[3] $\int_{\mathcal{X}} c_1(\mathcal{X}) \wedge c_1(\mathcal{X}) = 9 - N$ and $\int_{\mathcal{X}} c_2(\mathcal{X}) = N + 3$ for $\mathcal{X} = \tilde{\mathcal{B}}_N$.

Note that this bilinear form is not symmetric, the antisymmetric part is given by the determinant

$$\langle \mathcal{E}_1, \mathcal{E}_2 \rangle - \langle \mathcal{E}_2, \mathcal{E}_1 \rangle = - \begin{vmatrix} \mathrm{r}_1 & \mathrm{r}_2 \\ \mathrm{d}_{\Sigma_1} & \mathrm{d}_{\Sigma_2} \end{vmatrix} . \tag{2.11}$$

With this bilinear form $K(\widetilde{\mathcal{B}}_N)$ has signature $(N+1, 2)$. This scalar product was also considered in [23] where it was shown that for certain manifolds the matrix of scalar products of exceptional sheaves is the same as the soliton counting matrix [24] of the corresponding Landau-Ginzburg theory. We plan to explore the relation between the exceptional sheaves on Del Pezzos, solitons of the corresponding $\mathcal{N} = 2$ Landau-Ginzburg theories and string junctions living on affine 7-brane backgrounds further in the future.

2.2 Probe theory in IIB

Four-dimensional $\mathcal{N} = 2$ theories can be realized as the world-volume theory of a D3-brane probe in the vicinity of some 7-branes of IIB string theory. The algebra of the 7-branes appears as the global symmetry in the 4d field theory. The states arise from strings and string junctions [25] stretched between the D3 and (some of) the 7-branes and are characterized by the charges of each 7-brane. Viewing the setup as an F-theory compactification on K3, these states have the following geometrical interpretation. The position of the D3-brane singles out a fixed elliptic fiber E_* and the strings/junctions with a $\binom{p}{q}$ string segment ending on the D3-brane can be viewed as curves in the K3 whose boundary wraps the (p, q)-cycle of E_*. In the dual M-theory picture the selected elliptic fiber is wrapped by an M5-brane and the states arise from M2 branes wrapping the curves in K3 which end on the M5-brane [26].

The low energy field theory on the D3-brane in the vicinity of certain configurations of m 7-branes is insensitive to the remaining $24 - m$ 7-branes.[4] For the sake of simplicity we might replace the K3 in our arguments by an elliptically fibered *noncompact* Ricci-flat manifold $\mathcal{M}_m$ containing these m singular fibers only. Then the spectrum of the four dimensional probe theory is characterized by the curves of (the elliptically fibered) $\mathcal{M}_m$ with boundary on the selected elliptic fiber E_*, or the relative homology $H_2(\mathcal{M}_m, E_*)$. The natural norm on this lattice is given by the self-intersection of the homology elements. The BPS states correspond to the holomorphic curves, which satisfy

$$^{\#}(\mathcal{C} \cdot \mathcal{C}) = 2g - 2 + b \,, \tag{2.12}$$

b and g being the number of boundary components on the E_* and the genus of the curve respectively.

[4] It was shown in [30] that the only 7-brane backgrounds which allow such a decoupling are the ones with elliptic or parabolic monodromy.

Let us now specify the particular 7-brane backgrounds of our interest. The well-known $\mathbf{E_N}$ Kodaira singularities ($N < 9$) can be understood in terms of $N+2$ 7-branes, of which at most N may be mutually local. A string (junction) state can be identified by specifying its charge with respect to each 7-brane (linking number [27] or invariant charge [28]) and thus is an element of an $N+2$ dimensional lattice. To find the theories which are dual to the ones presented in the previous section, we need a lattice of one more dimension. As it was pointed out in [29], the relevant 7-brane background is obtained by adding an extra 7-brane to the $\mathbf{E_N}$ configuration so that it becomes $\widehat{\mathbf{E}}_\mathbf{N}$, which was studied in detail in [31, 32, 33] and is summarized in the following subsections. The junction lattice of $\widehat{\mathbf{E}}_\mathbf{N}$, $\mathcal{J}^{2,N+1}$, is $N+3$ dimensional and is of signature $(2, N+1)$.

2.2.1 $\widehat{\mathbf{E}}_{\mathbf{N<9}}$

We consider the type-IIB background with $N+3$ non-local 7-branes of the configuration $\widehat{\mathbf{E}}_{N<9}$. The elliptic fibration of the corresponding F-theory manifold, $\widehat{\mathcal{E}}_N$ is characterized by the monodromy around the singular fibers which is encoded in the 7-brane charges [34]. Denote a $[p,q]$ 7-brane as $\mathbf{X_{[p,q]}}$, with corresponding inverse monodromy[5] $K_{p,q} = \left(\begin{smallmatrix} 1+pq & -p^2 \\ q^2 & 1-pq \end{smallmatrix}\right)$. Then $\widehat{\mathcal{E}}_N$ is defined in terms of the following 7-brane configuration:[6]

$$(\mathbf{X_{[1,0]}})^\mathbf{N}\, \mathbf{X_{[6,-1]}}\mathbf{X_{[-3,1]}} \quad \mathbf{X_{[0,1]}} \qquad\qquad K(\widehat{\mathcal{E}}_N) = \begin{pmatrix} 1 & 9-N \\ 0 & 1 \end{pmatrix}, \tag{2.13}$$

$K(\widehat{\mathcal{E}}_N)$ being the overall monodromy. If we remove the $\mathbf{X_{[0,1]}}$-brane, the remaining $N+2$ 7-branes can be collapsed to the $\mathbf{E_N}$ Kodaira-singularity with overall monodromy and associated binary quadratic form [32]:

$$K(\mathcal{E}_N) = \begin{pmatrix} 1 & 9-N \\ -1 & N-8 \end{pmatrix} \qquad f_{E_N}(p,q) = \frac{p^2}{9-N} + pq + q^2\,. \tag{2.14}$$

Let us consider $\mathbf{E_N}$ first. In a suitable basis [28] a junction can be written as $\mathbf{J} = \sum_{i=1}^N \lambda_i \boldsymbol{\omega}^i + p\boldsymbol{\omega}^p + q\boldsymbol{\omega}^q$, and the self-intersection form on the homology lattice $H_2(\mathcal{E}_N, E_*)$ factorizes such that the curve $\mathbf{J}$ with boundary wrapping the (p,q)-cycle of the E_* has norm

$$(\mathbf{J},\mathbf{J}) = -\lambda_{E_N}^2 + f_{E_N}(p,q)\,, \tag{2.15}$$

where λ is a vector of the E_N weight lattice. Back to $\widehat{\mathcal{E}}_N$, the addition of the "affining" $\mathbf{X_{[0,1]}}$ 7-brane gives rise to one more basis element, which we call $\delta^{(-1,0)}$ and choose

[5] $K_{p,q}$ is the $\mathrm{SL}(2,\mathbb{Z})$ action felt by a string as it crosses the branch cut of the 7-brane.

[6] This configuration is identical to $\widehat{\widehat{\mathbf{E}}}_\mathbf{N}$ of [33] up to an overall transformation with $T^4 \in \mathrm{SL}(2,\mathbb{Z})$.

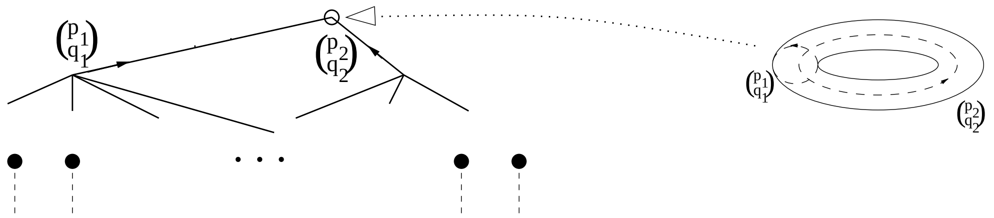

Figure 1: Two curves (junctions) whose boundary wraps intersecting cycles of the selected elliptic fiber.

to be a closed $\binom{-1}{0}$ string (oriented counterclockwise) encircling all of the 7-branes:

$$
\begin{aligned}
\mathbf{J} &= \sum_{i=1}^{N} \lambda_i \boldsymbol{\omega}^i + p\boldsymbol{\omega}^p + q\boldsymbol{\omega}^q + n\delta^{(-1,0)}\,, \\
(\mathbf{J},\mathbf{J}) &= -\lambda_{E_N}^2 + 2nq + f_{E_N}(p,q)\,.
\end{aligned}
\tag{2.16}
$$

In this case the norm does not factorize as in (2.15); the first two terms, however, can be regarded as the norm of a weight vector in the affine lattice $\widehat{E}_N$.

Eqns. (2.15) and (2.16) determine the self-intersection number of a curve in $\widehat{\mathcal{E}}_N$ with boundary on the selected fiber. There is some ambiguity in extending this formula to mutual intersection between curves whose boundary wraps different cycles of the torus, which is resolved by specifying the contribution of the intersecting boundaries of two curves depicted in figure 1.

Linearity of the intersection number requires that if $\mathbf{J}_1$ and $\mathbf{J}_2$ wrap the (p_1,q_1) and (p_2,q_2) cycle of the T^2 respectively, then the contribution from the boundary is

$$
(\mathbf{J}_1,\mathbf{J}_2)_{\text{boundary}} = \alpha \begin{vmatrix} p_1 & p_2 \\ q_1 & q_2 \end{vmatrix}, \qquad (\mathbf{J}_2,\mathbf{J}_1)_{\text{boundary}} = (1-\alpha) \begin{vmatrix} p_1 & p_2 \\ q_1 & q_2 \end{vmatrix}. \tag{2.17}
$$

In [28] the contribution was postulated to be symmetric ($\alpha = 1/2$), which led to fractional intersection numbers in general. This is not suitable for us because the metric on the lattice of the dual theory is manifestly integral. We shall utilize the simplest integral choice: $\alpha = 1$, so that $(\mathbf{J}_1,\mathbf{J}_2)_{\text{boundary}} = p_1 q_2 - q_1 p_2$ and $(\mathbf{J}_2,\mathbf{J}_1)_{\text{boundary}} = 0$. Together with (2.16) this uniquely fixes the intersection matrix on the junction lattice which is straightforward to determine.

Summary: a 3-brane parallel to the singular fibers of an F-theory compactification on a $\widehat{\mathcal{E}}_N$ manifold realizes a $d=4$, $\mathcal{N}=2$ theory whose states are characterized by an $N+3$ dimensional charge lattice equipped with following intersection bilinear form in the basis $\{\alpha_i | i=1,\ldots,N; \omega^p, \omega^q, \delta^{(-1,0)}\}$:

$$
\begin{pmatrix}
-E_N & & & \\
& \frac{1}{9-N} & 1 & 0 \\
& 0 & 1 & 1 \\
& 0 & 1 & 0
\end{pmatrix}, \tag{2.18}
$$

where E_N is the Cartan matrix of the corresponding Lie algebra.

2.2.2 $\widehat{\mathbf{E}}_{\mathbf{9}}$

Going beyond $N = 8$ in the series of the 7-brane configurations of (2.13) we encounter $\widehat{\mathcal{E}}_9$ which is special from numerous aspects. The overall monodromy is trivial and as a consequence, it admits two linearly independent closed strings encircling the 7-branes. It is useful to visualize the configuration in the following way:

$$\mathbf{X}_{[\mathbf{1},\mathbf{0}]}\left((\mathbf{X}_{[\mathbf{1},\mathbf{0}]})^{\mathbf{8}}\mathbf{X}_{[\mathbf{6},-\mathbf{1}]}\mathbf{X}_{[-\mathbf{3},\mathbf{1}]}\right)\mathbf{X}_{[\mathbf{0},\mathbf{1}]} = \mathbf{X}_{[\mathbf{1},\mathbf{0}]}\left(\mathbf{E}_{\mathbf{8}}\right)\mathbf{X}_{[\mathbf{0},\mathbf{1}]}\,, \tag{2.19}$$

and think about it as being "doubly affined" [33]. One possible basis for the lattice of junctions being supported on this configuration is

$$\begin{aligned}
\mathbf{J} &= \sum_{i=1}^{8} \lambda_i \boldsymbol{\omega}^i + p\boldsymbol{\omega}^p + q\boldsymbol{\omega}^q + n_1\,\delta^{(-1,0)} + n_2\,\delta^{(0,1)}\,, && (2.20)\\
(\mathbf{J},\mathbf{J}) &= -\lambda_{E_8}^2 + 2n_1 q + 2n_2 p + f_{E_8}(p,q)\\
&= -\lambda_{E_8}^2 + 2n_1 q + 2n_2 p + p^2 + pq + q^2\,, && (2.21)
\end{aligned}$$

with intersection matrix in the basis $\{\alpha_i | i = 1, \ldots, 8; \omega^p, \omega^q, \delta^{(-1,0)}, \delta^{(0,1)}\}$ is

$$\begin{pmatrix} E_8 & & & & \\ & 1 & 1 & 0 & 1 \\ & 0 & 1 & 1 & 0 \\ & 0 & 1 & 0 & 0 \\ & 1 & 0 & 0 & 0 \end{pmatrix}. \tag{2.22}$$

2.3 3-cycles and string junctions

Mirror symmetry relates type-IIA string theory compactified on a Calabi-Yau threefold M to type IIB on the mirror manifold W [35]. It follows from the interpretation of mirror symmetry as T-duality [36, 37] that the even cohomology classes of M are mapped to the odd cohomology classes of W and therefore the complexified Kähler structure parameters of M are exchanged with the complex structure parameters of W [38]. Vector bundles with characteristic classes represented by the even cohomology classes map to 3-cycles dual to the odd cohomology classes [39].

2.3.1 The mirror of a Calabi-Yau threefold containing $\mathcal{B}_9$

A Calabi-Yau threefold, M, containing $\mathcal{B}_9$ can be described as a double elliptic fibration over $\mathbb{P}^1$ [21],

$$y_i^2 = x_i^3 + f_{4,i}(z)\,x_i + g_{6,i}(z)\,, \qquad i = 1, 2\,, \tag{2.23}$$

where z is the coordinate on the $\mathbb{P}^1$. The total space of each fibration over the sphere defines a $\mathcal{B}_9$ surface. As shown in [21], this CY threefold can also be obtained by

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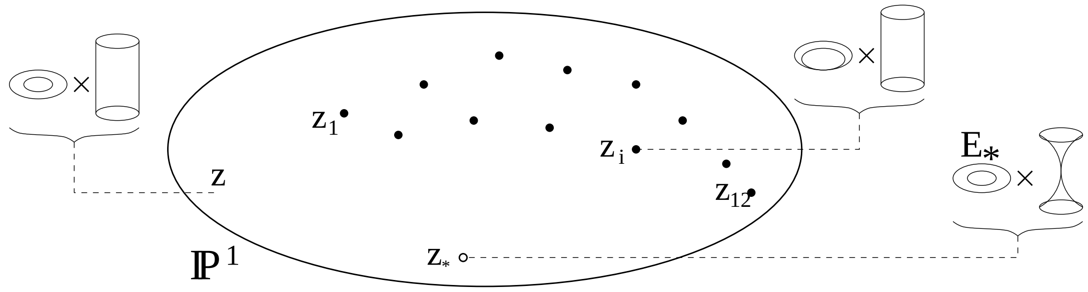

Figure 2: The non-compact mirror Calabi-Yau W, as an $S_c^1 \times \mathbb{R}$ fibration over the base of the elliptically fibered $\mathcal{B}_9$. At the points z_i different cycles of the elliptic fiber of $\mathcal{B}_9$ are shrinking while at z_* the S_c^1 shrinks.

resolving the singularities of a $\mathbb{Z}_2 \times \mathbb{Z}_2$ orbifold of $T^2 \times T^2 \times T^2$. If $\xi_{1,2,3}$ are the complex coordinates on the three tori then the orbifold action is given by [21]

$$(\xi_1, \xi_2, \xi_3) \to \left(\xi_1 + \frac{1}{2}, \xi_2, -\xi_3\right) \qquad \text{and} \qquad (\xi_1, \xi_2, \xi_3) \to \left(\xi_1, \xi_2 + \frac{1}{2}, -\xi_3\right) \tag{2.24}$$

and the third torus becomes the $\mathbb{P}^1$ after the identification. The holomorphic 3-form is $\Omega^{(3)} = dz \frac{dx_1}{y_1} \frac{dx_2}{y_2}$ with z being the coordinate on the $\mathbb{P}^1$. To obtain the non-compact CY-threefold containing $\mathcal{B}_9$, we decompactify the fiber of one of the elliptic fibrations by taking its area to infinity i.e. we consider the limit of the complexified Kähler parameter $B + iA \equiv \rho \to i\infty$.

The mirror threefold W is obtained by performing T-duality on one of the cycles of each torus. The decompactification limit of M then maps to the limit of W when the complex structure parameter τ of the elliptic fibration goes to $i\infty$ which in effect decompactifies one of the cycles of the torus. The local model for this is given by

$$y_2^2 - x_2^2 = (z - z_*)\,. \tag{2.25}$$

Since we want the degenerate complex structure limit for the entire elliptic fibration, we adopt the above model globally over $\mathbb{P}^1$. The structure of W then is that of a $T^2 \times S_c^1 \times \mathbb{R}$ fibration over a $\mathbb{P}^1$ and the holomorphic 3-form becomes

$$\Omega^{(3)} = dz \frac{dx_1}{y_1} \frac{dx_2}{x_2} = \frac{dx_2}{x_2} \Omega^{(2)}, \tag{2.26}$$

where $\Omega^{(2)}$ is the holomorphic 2-form on $\mathcal{B}_9$. Since the canonical bundle of $\mathcal{B}_9$ is non-trivial this holomorphic two form has zeros or poles at the 2-cycle dual to the first Chern class of the canonical bundle. The total space can be visualized as a double fibration over $\mathbb{P}^1$: the T^2 fibration constitutes a $\mathcal{B}_9$ surface and a cylinder is also fibered over its base. At one point $z_* \in \mathbb{P}^1$ the nontrivial cycle of the $S_c^1 \times \mathbb{R}$ shrinks to zero size. We denote the elliptic fiber of $\mathcal{B}_9$ above this point by E_*.

2.3.2 3-Cycles in W

Mirror symmetry maps the even homology of M to the odd homology of W. Since W is simply connected the only odd homology elements of W are the 3-cycles of the following type [21]:

- $S_c^1 \times S^2$: Here $S^2 = C$ is a curve in $\mathcal{B}_9$ and S_c^1 is the non-trivial cycle of the $S^1 \times \mathbb{R}$ fibration. There are eight such cycles on which the holomorphic 3-form $\Omega^{(3)}$ is non-zero: $S_c^1 \times C_i$, $i = 1, \ldots, 8$ with C_i given in (2.3) corresponding to the roots of the E_8 root lattice embedded in $H_2(\mathcal{B}_9)$.

- S^3: There are two such (linearly independent) cycles. They are formed by $S_c^1 \times S^1$ fibered over intervals $\mathcal{I}_{1,2}$ such that S^1 shrinks on one end of the interval and S_c^1 shrinks at the other side.

- T^3: There are again two (linearly independent) cycles of this type. These are formed by S_c^1 in the $S_c^1 \times \mathbb{R}$ fibration and a torus in the $\mathcal{B}_9$. The torus in the $\mathcal{B}_9$ is formed from the circle surrounding the position of the degenerate fibers on the base and a 1-cycle of the elliptic fiber.

2.3.3 String junctions from 3-cycles

Let $\pi : \mathcal{B}_9 = \mathcal{X} \to \mathbb{P}^1$ be an elliptic fibration. We denote by $E_* = \pi^{-1}(z_*)$, as discussed before, a fixed non-degenerate elliptic fiber. The section $e : \mathbb{P}^1 \to \mathcal{X}$ is such that $e(\mathbb{P}^1) = B$ with $B^2 = -1$. The three-cycles in the non-compact Calabi-Yau threefold W are of the form $\mathcal{C} \times S_c^1$, where $\mathcal{C}$ is a curve in $\mathcal{B}_9$ with a boundary such that $\mathcal{C} \cap E_* = \partial\mathcal{C} \in H_1(E_*, \mathbb{Z})$. With any such curve $\mathcal{C} \in \mathcal{X}$ we can associate a junction $\mathbf{J}_\mathcal{C}$ living on the base B,

$$\mathbf{J}_\mathcal{C} = e(\pi(\mathcal{C}))\,. \tag{2.27}$$

If $\partial\mathcal{C} = 0$, the corresponding junction has no asymptotic charge and has support only on the 7-branes, the positions of the degenerate fibers. When $\partial\mathcal{C} = (p, q) \in H_1(E_*, \mathbb{Z})$ the corresponding junction has asymptotic charge (p, q) on the D3-brane, the position of E_* on the base.

We summarize the mirror symmetry map between the homology of $\mathcal{B}_9$ and the 3-cycles of the mirror Calabi-Yau W in table 1. $\mathcal{I}_1$ and $\mathcal{I}_2$ represent curves on the base from z_1 and z_{12} to z_* respectively. δ is the path encircling the 7-branes. S^1 and S_D^1 are the two basis 1-cycles of the elliptic fiber.

Table 2 shows the Del Pezzo surfaces and the corresponding dual 7-brane configurations.

$\mathbf{H}_*(\mathcal{B}_\mathbf{9})$	$\mathbf{H_3(W)}$	$\mathcal{J}^{2,10}$	$A(\Sigma) = \int_\Sigma \Omega^{(2)}$
$C_i, \quad i = 1, \ldots, 8$	$S^2 \times S^1_c = C_i \times S^1_c$	C_i	$m_i = \int_{C_i} dz \, \frac{dx}{y}$
B	$S^3 = \mathcal{I}_1 \times (S^1 \times S^1_c)$	$\mathbf{a}_1 = \mathbf{x}_{[1,0]}$	$\phi = \int_{z_1}^{z_*} dz \oint_{[1,0]} \frac{dx}{y}$
F	$T^3 = \delta \times S^1 \times S^1_c$	$\delta^{(0,1)}$	$\tau = \oint_\delta dz \oint_{[0,1]} \frac{dx}{y}$
$\mathcal{B}_9$	$S^3 = \mathcal{I}_2 \times (S^1_D \times S^1_c)$	$\mathbf{x}_{[0,1]}$	$\phi_D = \int_{z_{12}}^{z_*} dz \oint_{[0,1]} \frac{dx}{y}$
0–cycle	$T^3 = \delta \times S^1_D \times S^1_c$	$\delta^{(-1,0)}$	$\tau_D = \oint_\delta dz \oint_{[-1,0]} \frac{dx}{y}$

Table 1: Homology cycles of $\mathcal{B}_9$ and their mirror images.

Complex surface	Brane Configuration	Algebra
$\tilde{\mathcal{B}}_0 = \mathbb{P}^2$	$\mathbf{X}_{[\mathbf{6},-\mathbf{1}]}\mathbf{X}_{[-\mathbf{3},\mathbf{1}]}\mathbf{X}_{[\mathbf{0},\mathbf{1}]}$	$\widehat{E}_0$
$\widetilde{\mathcal{B}}_1 = \mathbb{P}^2 \# \bar{\mathbb{P}}^2$	$\mathbf{X}_{[\mathbf{6},-\mathbf{1}]}\mathbf{X}_{[-\mathbf{3},\mathbf{1}]}\mathbf{X}_{[\mathbf{0},\mathbf{1}]}$	$\widehat{\tilde{E}}_1 = \widehat{\mathrm{u}(1)}$
$\mathcal{B}_1 = \mathbb{P}^1 \times \mathbb{P}^1$	$\mathbf{X}_{[\mathbf{1},-\mathbf{1}]}\mathbf{X}_{[\mathbf{1},\mathbf{1}]}\mathbf{X}_{[\mathbf{1},-\mathbf{1}]}\mathbf{X}_{[\mathbf{1},-\mathbf{1}]}$	$\widehat{E}_1 = \widehat{\mathrm{su}(2)}$
$\widetilde{\mathcal{B}}_{N>1} = \begin{cases} \mathbb{P}^2 \# \underbrace{\bar{\mathbb{P}}^2 \# \ldots \# \bar{\mathbb{P}}^2}_{N} \\ (\mathbb{P}^1 \times \mathbb{P}^1) \# \underbrace{\bar{\mathbb{P}}^2 \# \ldots \# \bar{\mathbb{P}}^2}_{N-1} \end{cases}$	$\begin{cases} \mathbf{A}^N \mathbf{X}_{[\mathbf{6},-\mathbf{1}]}\mathbf{X}_{[-\mathbf{3},\mathbf{1}]}\mathbf{X}_{[\mathbf{0},\mathbf{1}]} \\ \mathbf{A}^{N-1}(\mathbf{X}_{[\mathbf{1},-\mathbf{1}]}\mathbf{X}_{[\mathbf{1},\mathbf{1}]})^\mathbf{2} \end{cases}$	$\widehat{E}_N$

Table 2: 7-brane configurations dual to Del Pezzo surfaces.

3. String junctions and vector bundles

3.1 $\mathcal{B}_9$ and $\widehat{\mathbf{E}}_\mathbf{9}$

We map the equivalence classes of sheaves on $\mathcal{B}_9$ to string junctions with support on the $\widehat{\mathbf{E}}_9$ 7-brane configuration. Recall that in [29] the map between curves and junctions of zero q charge living on $\mathbf{E_9}$ 7-brane configuration was given. According to that map a curve

$$\Sigma = \sum_{i=1}^{8} \lambda_i \omega^i + \mathrm{d}_\Sigma (B + F) + \mathrm{c}F \in H_2(\mathcal{B}_9, \mathbb{Z}) \,, \tag{3.1}$$

corresponds to a family of junctions

$$\mathbf{J}_\Sigma(m) \equiv \sum_{i=1}^{8} \lambda_i \boldsymbol{\omega}^i + \mathrm{d}_\Sigma \boldsymbol{\omega}^p + \mathrm{c}\delta^{(0,1)} + m\delta^{(-1,0)} \,, \qquad m \in \mathbb{Z} \,. \tag{3.2}$$

Different values of m correspond to different bundles on Σ. We denote by $\mathcal{O}_\Sigma(m)$ a (torsion sheaf whose restriction to its support is a) bundle on Σ with $\mathrm{ch}(\mathcal{O}_\Sigma(m)) = (0, \Sigma, m + \frac{1}{2}\mathrm{d}_\Sigma)$. A D2-brane wrapped on B maps to a D4-brane wrapped on $\mathcal{B}_9$ after

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T-duality, which we will interpret as an $\mathrm{SL}(2,\mathbb{Z})$ transformation by S. Therefore we require that the map between the junctions and bundles should satisfy the following conditions:

- The rank of the bundle $\mathcal{F}$ is the q-charge of the corresponding junction.
- The degree d_Σ of the first Chern class Σ of the bundle, is the p-charge of the junction.
- $\langle \mathcal{F}, \mathcal{F} \rangle = -^{\#}(\mathbf{J}_\mathcal{F} \cdot \mathbf{J}_\mathcal{F})$.

By equating the two scalar products and requiring that they be equal for all $(r, \mathrm{d}_\Sigma) = (q, p)$ we get a unique map between the bundle data and the junction data. It follows that a bundle $\mathcal{F}$ with

$$\mathrm{ch}(\mathcal{F}) = (\mathrm{r}\,, \Sigma\,, \mathrm{k})\,, \qquad \Sigma = \sum_{i=1}^{8} \lambda_i \omega^i + \mathrm{d}_\Sigma (B + F) + \mathrm{c} F\,, \tag{3.3}$$

corresponds to the junction

$$\mathbf{J}_\mathcal{F} = \sum_{i=1}^{8} \lambda_i \boldsymbol{\omega}^i + \mathrm{d}_\Sigma \boldsymbol{\omega}^p + c\delta^{(0,1)} + \mathrm{r}\boldsymbol{\omega}^q - \left(\mathrm{r} + \mathrm{k} + \frac{1}{2}\mathrm{d}_\Sigma\right) \delta^{(-1,0)}\,. \tag{3.4}$$

3.2 $\mathcal{B}_N$ and $\widehat{\mathbf{E}}_\mathbf{N}$

The map between $K(\mathcal{B}_N)$ and $\widehat{\mathbf{E}}_\mathbf{N}$ for $N < 9$ follows from the above map. Blowing down an exceptional curve corresponds to removing a $\mathbf{X}_{[\mathbf{1},\mathbf{0}]}$ 7-brane of the $\widehat{\mathbf{E}}_\mathbf{9}$ 7-brane configuration. By this process we not only decouple the string junction with support on that brane but we also lose the $\delta^{(0,1)}$ string junction. It then follows from (3.4) that a class $\mathcal{F} \in K(\mathcal{B}_N)$ such that

$$\mathrm{ch}(\mathcal{F}) = (\mathrm{r}\,, \Sigma\,, \mathrm{k})\,, \qquad \Sigma = \sum_{i=1}^{N} \lambda_i \omega^i - \frac{\mathrm{d}_\Sigma}{9-N} K_{\mathcal{B}_N} \in H_2(\mathcal{B}_N, \mathbb{Z})\,, \tag{3.5}$$

corresponds to

$$\mathbf{J}_\mathcal{F} = \sum_{i=1}^{N} \lambda_i \omega^i + \mathrm{d}_\Sigma \omega^p + \mathrm{r}\omega^q - \left\{\mathrm{r} + \mathrm{k} + \frac{1}{2}\mathrm{d}_\Sigma\right\} \delta^{(-1,0)}\,. \tag{3.6}$$

where $\mathrm{r} + \mathrm{k} + \frac{1}{2}\mathrm{d}_\Sigma \equiv \chi(\mathcal{F})$ is the Euler-Poincaré characteristic of $\mathcal{F}$. $K(\mathcal{B}_N)$ is an abelian group, the inverse of $\mathcal{F}$ is $-\mathcal{F}$ and the associated junction is $-\mathbf{J}_\mathcal{F}$. With this identification we get

$$\langle \mathcal{F}_1, \mathcal{F}_2 \rangle = -^{\#}(\mathbf{J}_{\mathcal{F}_1} \cdot \mathbf{J}_{\mathcal{F}_2})\,. \tag{3.7}$$

3.3 Genus of the junction and dimension of the moduli space

Let $\mathcal{F}$ be a stable holomorphic vector bundle on $\mathcal{B}_9$ with $\mathrm{ch}(\mathcal{F}) = (\mathrm{r}, \Sigma, \mathrm{k})$ such that $\gcd(\mathrm{r}, \mathrm{d}_\Sigma) = 1$. We denote the corresponding special lagrangean 3-cycle in the mirror Calabi-Yau and the BPS junction by $C_\mathcal{F}$ and $\mathbf{J}_\mathcal{F}$ respectively. The dimension of the moduli space $\mathcal{M}(\mathrm{r}, \Sigma, \mathrm{k})$ of the vector bundle is $-\langle \mathcal{F}, \mathcal{F}\rangle + 1$ while the moduli space is empty if $\langle \mathcal{F}, \mathcal{F}\rangle > 1$ [41, 40]. From the correspondence with junctions we see that [42]

$$\begin{aligned} \dim \mathcal{M}(\mathrm{r}, \Sigma, \mathrm{k}) &= {}^\#(\mathbf{J}_\mathcal{F} \cdot \mathbf{J}_\mathcal{F}) + 1 \\ &= 2g - 2 + \gcd(\mathrm{r}, \mathrm{d}_\Sigma) + 1 = 2g\,, \end{aligned} \tag{3.8}$$

where g is the genus of the curve associated with the junction. This is in agreement with Vafa's conjecture [39] that the map between the 3-cycles and the vector bundles should be such that

$$H^i(\mathrm{End}\ \mathcal{F}) = H^i(C_\mathcal{F}, W)\,. \tag{3.9}$$

It follows from the above identification that

$$\begin{aligned} \langle \mathcal{F}, \mathcal{F}\rangle &\equiv \sum_{i=0}^{2} (-1)^i \dim \mathrm{Ext}^i(\mathcal{F}, \mathcal{F}) \equiv \sum_{i=0}^{2} (-1)^i \dim H^i(\mathrm{End}\ \mathcal{F}) \\ &= \sum_{i=0}^{2} (-1)^i \dim H^i(C_\mathcal{F}, W) = \sum_{i=0}^{2} (-1)^i b^i \\ &= \chi(C_\mathcal{F}) + 1 \end{aligned} \tag{3.10}$$

and therefore

$$\begin{aligned} \dim \mathcal{M}(\mathrm{r}, \Sigma, \mathrm{k}) &= -\langle \mathcal{F}, \mathcal{F}\rangle + 1 \\ &= -\chi(C_\mathcal{F}) = 2g\,. \end{aligned} \tag{3.11}$$

We summarize the results of this section in table 3. $\mathcal{O}_X$ is the structure sheaf of the manifold X and is the trivial rank one bundle corresponding to a D4-brane wrapped on X. $\mathcal{O}_\Sigma(m)$ and $\mathcal{O}_x$ are the torsion sheaf and the skyscraper sheaf respectively.

D-branes	$\mathcal{F}$	$\mathrm{ch}(\mathcal{F})$	string junction
D4-brane	$\mathcal{O}_{\widetilde{\mathcal{B}}_N}$	$(1, 0, 0)$	$\boldsymbol{\omega}^q - \delta^{(-1,0)}$
D2-brane + mD0-branes	$\mathcal{O}_\Sigma(-m), m \in \mathbb{Z}_{\geq 0}$	$(0, \Sigma, \frac{1}{2}\mathrm{d}_\Sigma - m)$	$\mathbf{J}_\Sigma(m - \mathrm{d}_\Sigma)$
D0-brane	$\mathcal{O}_x, x \in \mathcal{B}_N$	$(0, 0, -1)$	$\delta^{(-1,0)}$

Table 3: The duality map between IIA branes and string junctions.

3.4 Fourier-Mukai transform and $\mathrm{SL}(2,\mathbb{Z})$

In this section we will show that a Fourier-Mukai transformation [16] on $\mathcal{B}_9 = X$ can be identified with an $\mathrm{SL}(2,\mathbb{Z})$ transformation by $S = \left(\begin{smallmatrix} 0 & -1 \\ 1 & 0 \end{smallmatrix}\right)$ on the dual 7-brane background $\widehat{\mathbf{E}}_{\mathbf{9}}$.

Let $\mathcal{F}$ be a complex of sheaves[7] on $\pi : X \to \mathbb{P}^1$ such that $\mathrm{ch}(\mathcal{F}) = (\mathrm{r}, \Sigma, \mathrm{k})$. The Fourier-Mukai transform $\mathbf{S}$ maps this to a complex of sheaves $\mathbf{S}(\mathcal{F})$ on $\widehat{X}$, where $\widehat{\pi} : \widehat{X} \to \mathbb{P}^1$ is the dual elliptic fibration. Let $\widehat{\mathbf{S}}$ be the Fourier-Mukai transform which maps $\widehat{\mathcal{G}}$, a complex of sheaves on $\widehat{X}$ with $\mathrm{ch}(\widehat{\mathcal{G}}) = (\widehat{\mathrm{r}}, \widehat{\Sigma}, \widehat{\mathrm{k}})$, to $\widehat{\mathbf{S}}(\widehat{\mathcal{G}})$, a complex of sheaves on X. The Chern classes of these complexes are given by [16]

$$\begin{aligned}
\mathrm{ch}_0(\mathbf{S}(\mathcal{F})) &\equiv \widehat{\mathrm{r}}' = \mathrm{d}_\Sigma \,, \\
\mathrm{ch}_1(\mathbf{S}(\mathcal{F})) &\equiv \widehat{\Sigma}' = -\mathrm{w}(\Sigma) + (\mathrm{d}_\Sigma - \mathrm{r})B + \left\{ \mathrm{k} + {}^{\#}(\Sigma \cdot B) + \frac{1}{2}\mathrm{d}_\Sigma \right\} F \,, \\
\mathrm{ch}_2(\mathbf{S}(\mathcal{F})) &\equiv \widehat{\mathrm{k}}' = -\mathrm{d}_\Sigma - {}^{\#}(\Sigma \cdot B) + \frac{1}{2}\mathrm{r} \,,
\end{aligned} \tag{3.12}$$

and

$$\begin{aligned}
\mathrm{ch}_0(\widehat{\mathbf{S}}(\widehat{\mathcal{G}})) &\equiv \mathrm{r}' = \mathrm{d}_{\widehat{\Sigma}} \,, \\
\mathrm{ch}_1(\widehat{\mathbf{S}}(\widehat{\mathcal{G}})) &\equiv \Sigma' = \mathrm{w}^{-1}(\widehat{\Sigma}) - (\mathrm{d}_{\widehat{\Sigma}} + \widehat{\mathrm{r}})B + \left\{ \widehat{\mathrm{k}} - {}^{\#}(\widehat{\Sigma} \cdot B) - \frac{1}{2}\mathrm{d}_{\widehat{\Sigma}} \right\} F \,, \\
\mathrm{ch}_2(\widehat{\mathbf{S}}(\widehat{\mathcal{G}})) &\equiv \mathrm{k}' = -\mathrm{d}_{\widehat{\Sigma}} - {}^{\#}(\widehat{\Sigma} \cdot B) - \frac{1}{2}\widehat{\mathrm{r}} \,,
\end{aligned} \tag{3.13}$$

where $\mathrm{w} : H_2(X,\mathbb{Z}) \to H_2(\widehat{X},\mathbb{Z})$ is an automorphism induced by isomorphism between X and $\widehat{X}$. This automorphism is such that $\mathrm{w}(B) = B$ and $\mathrm{w}(F) = F$, therefore it corresponds to a Weyl transformation on the E_8 root lattice, $\Gamma_{E_8} \subset H_2(X,\mathbb{Z})$.

Thus we see that

$$\begin{pmatrix} \mathrm{d}_{\widehat{\Sigma}'} \\ \widehat{\mathrm{r}}' \end{pmatrix} = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} \mathrm{d}_\Sigma \\ \mathrm{r} \end{pmatrix}, \qquad \begin{pmatrix} \mathrm{d}_{\Sigma'} \\ \mathrm{r}' \end{pmatrix} = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} \mathrm{d}_{\widehat{\Sigma}} \\ \widehat{\mathrm{r}} \end{pmatrix}, \tag{3.14}$$

and

$$\begin{aligned}
\mathrm{ch}(\widehat{\mathbf{S}}(\mathbf{S}(\mathcal{F}))) &= -\,\mathrm{ch}(\mathcal{F}) \\
\langle\, \widehat{\mathbf{S}}(\widehat{\mathcal{G}}), \widehat{\mathbf{S}}(\widehat{\mathcal{G}})\,\rangle &= \langle\, \mathbf{S}(\mathcal{F}), \mathbf{S}(\mathcal{F})\,\rangle = \langle \mathcal{F}, \mathcal{F} \rangle \,.
\end{aligned} \tag{3.15}$$

[7] String junctions related by branch cut moves are physically equivalent. To a given $(\mathrm{r}, \Sigma, \mathrm{k})$ there corresponds a family of string junctions related to each other by branch cut moves. Since as shown in [44] only the image of the complex in the K-theory group is physically relevant, this image is sufficient to construct a member of the corresponding family of string junctions. These aspects of string junctions and their relation with derived categories is under investigation.

If we denote the junction corresponding to $\mathcal{F}$ by $\mathbf{J}_{\mathcal{F}}$, then[8]

$$\mathbf{J}_{\mathcal{F}} = \sum_{i=1}^{8} \lambda_i^{\Sigma}\omega^i + \mathrm{d}_\Sigma \omega^p + \mathrm{r}\omega^q + {}^{\#}(\Sigma\cdot B)\delta^{(0,1)} - \left(\mathrm{r}+\mathrm{k}+\frac{1}{2}\mathrm{d}_\Sigma\right)\delta^{(-1,0)}\,,$$

$$\mathbf{J}_{\mathbf{S}(\mathcal{F})} = -\sum_{i=1}^{8} \mathrm{w}(\lambda_i^{\Sigma})\omega^i - \mathrm{r}\omega^p + \mathrm{d}_\Sigma\omega^q + \left\{\mathrm{r}+\mathrm{k}-\frac{1}{2}\mathrm{d}_\Sigma\right\}\delta^{(0,1)} + {}^{\#}(\Sigma\cdot B)\delta^{(-1,0)}\,,$$

and

$$\mathbf{J}_{\widehat{\mathcal{G}}} = \sum_{i=1}^{8} \lambda_i^{\widehat{\Sigma}}\omega^i + \mathrm{d}_{\widehat{\Sigma}}\omega^p + \widehat{\mathrm{r}}\omega^q + {}^{\#}(\widehat{\Sigma}\cdot B)\delta^{(0,1)} - \left(\widehat{\mathrm{r}}+\widehat{\mathrm{k}}+\frac{1}{2}\mathrm{d}_{\widehat{\Sigma}}\right)\delta^{(-1,0)}\,, \tag{3.16}$$

$$\mathbf{J}_{\widehat{\mathbf{S}}(\widehat{\mathcal{G}})} = \sum_{i=1}^{8} \mathrm{w}^{-1}(\lambda_i^{\widehat{\Sigma}})\omega^i - \widehat{\mathrm{r}}\omega^p + \mathrm{d}_{\widehat{\Sigma}}\omega^q + \left\{\widehat{\mathrm{r}}+\widehat{\mathrm{k}}+\frac{1}{2}\mathrm{d}_{\widehat{\Sigma}}\right\}\delta^{(0,1)} + \left\{{}^{\#}(\widehat{\Sigma}\cdot B)+\widehat{\mathrm{r}}\right\}\delta^{(-1,0)}\,.$$

It is straightforward to show that $\mathbf{J}_{\mathbf{S}(\mathcal{F})}$ and $\mathbf{J}_{\widehat{\mathbf{S}}(\widehat{\mathcal{G}})}$ are obtained from $\mathbf{J}_{\mathcal{F}}$ and $\mathbf{J}_{\widehat{\mathcal{G}}}$ respectively by a global $\mathrm{SL}(2,\mathbb{Z})$ transformation by $S = \left(\begin{smallmatrix} 0 & -1 \\ 1 & 0 \end{smallmatrix}\right)$ and branch cut moves inducing the E_8 Weyl transformation w. The branch cut moves, however, are different for $\mathbf{J}_{\mathbf{S}(\mathcal{F})}$ and $\mathbf{J}_{\widehat{\mathbf{S}}(\widehat{\mathcal{G}})}$ and correspond to different Weyl transformations of affine E_8 [43]. Therefore we see that the Fourier-Mukai transform is the S-duality transformation of the type-IIB 7-brane background $\widehat{\mathbf{E}}_{\mathbf{9}}$, and the sign ambiguity referred to in [44, 45] is required for the proper identification between junctions and vector bundles.

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